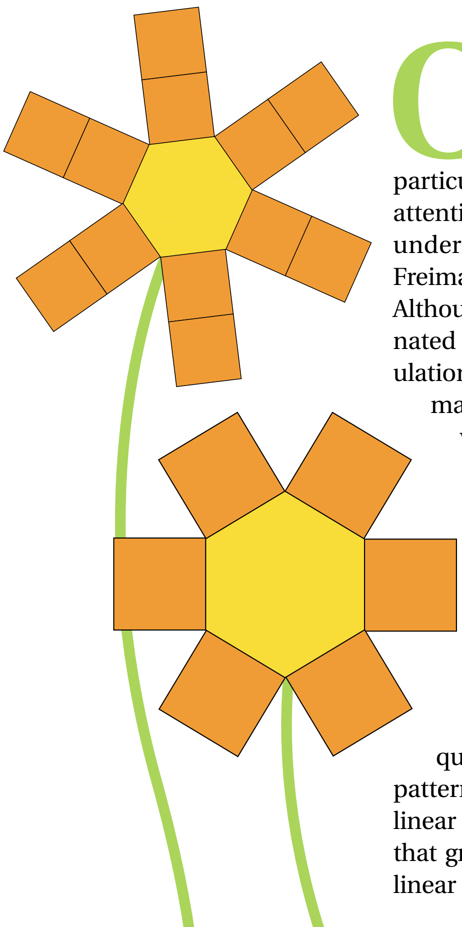
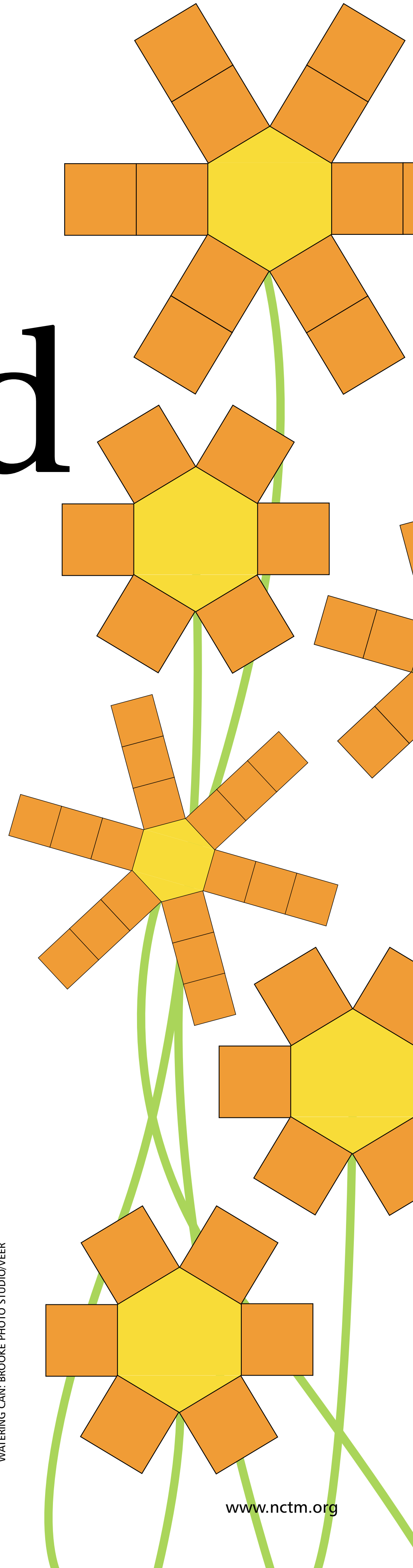
Growing Patterns:

Seeing beyond Counting

These three suggestions help sixth-grade students develop functional thinking in geometry.

By Kimberly A. Markworth

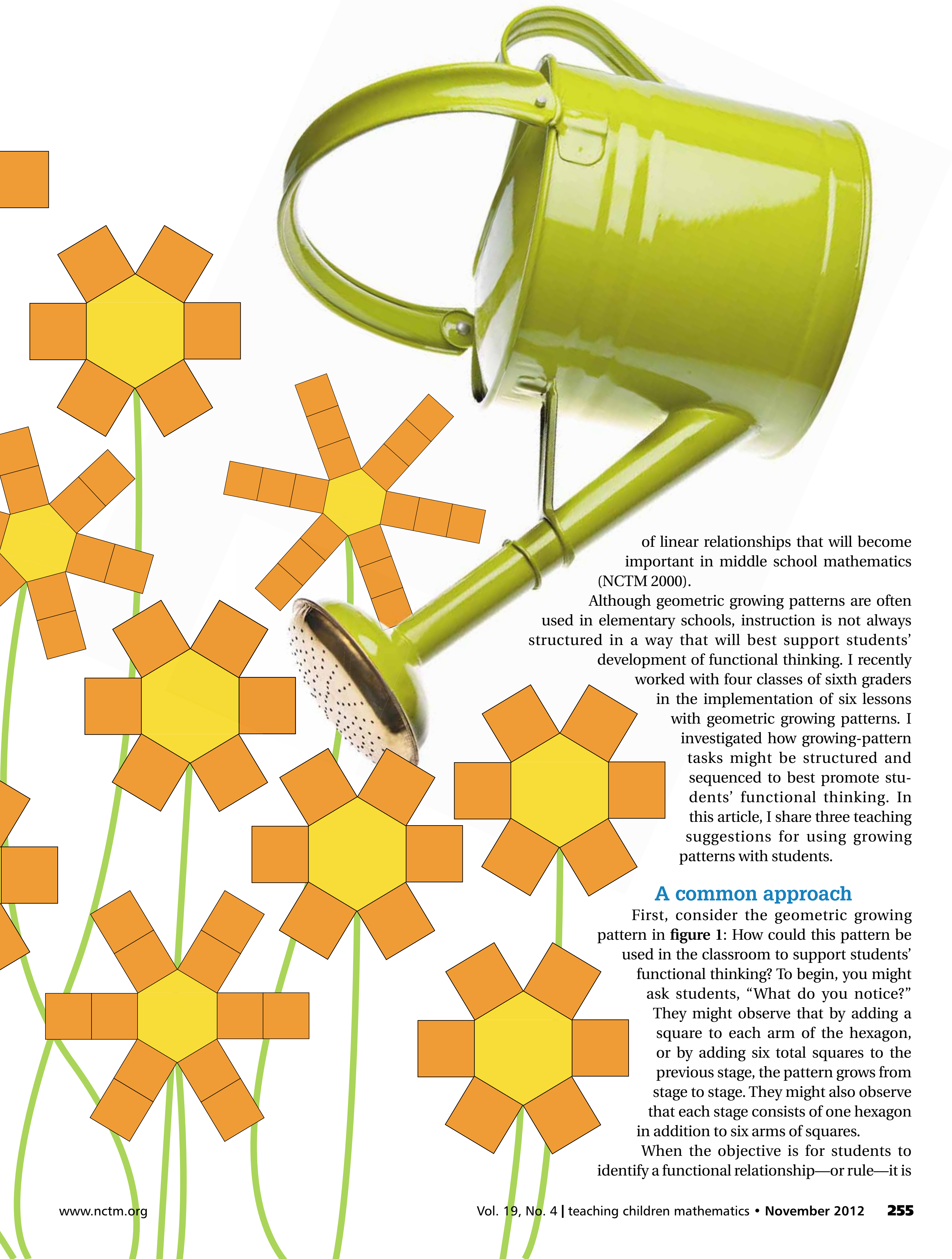




Over the past two decades, mathematical patterns have been acknowledged as important early components of children's development of algebraic reasoning (NCTM 2000). In particular, growing patterns have attracted significant attention as a context that helps students develop an understanding of functional relationships (Lee and Freiman 2006; Moss et al. 2006; Rivera and Becker 2009). Although traditional algebra curricula have been dominated by an approach that focuses on symbolic manipulation, solving equations, and simplifying expressions, many programs now use a functional approach in which the symbolic manipulations are couched in the context of functions and modeling of real-world situations (Kieran 2007).

The use of growing patterns in elementary schools provides experiences with algebra in a way that supports this shift in approach. Geometric growing patterns help students develop *functional thinking*; that is, "representational thinking that focuses on the relationship between two (or more) varying quantities" (Smith 2008, p. 143). Geometric growing patterns can represent both linear functions and nonlinear functions. However, geometric growing patterns that grow by a fixed number of parts in each stage—a linear relationship—provide a foundation for the study

WATERING CAN: BROOKE PHOTO STUDIO/VEER



of linear relationships that will become important in middle school mathematics (NCTM 2000).

Although geometric growing patterns are often used in elementary schools, instruction is not always structured in a way that will best support students' development of functional thinking. I recently worked with four classes of sixth graders in the implementation of six lessons with geometric growing patterns. I investigated how growing-pattern tasks might be structured and sequenced to best promote students' functional thinking. In this article, I share three teaching suggestions for using growing patterns with students.

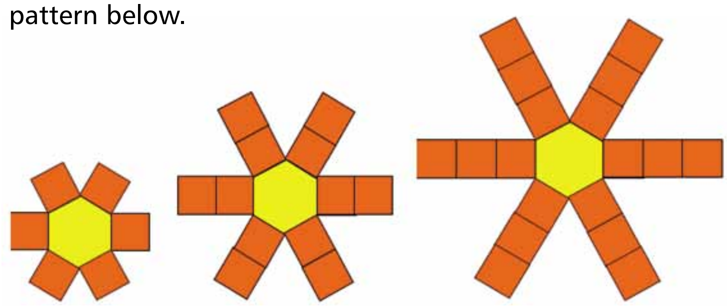
A common approach

First, consider the geometric growing pattern in **figure 1**: How could this pattern be used in the classroom to support students' functional thinking? To begin, you might ask students, "What do you notice?" They might observe that by adding a square to each arm of the hexagon, or by adding six total squares to the previous stage, the pattern grows from stage to stage. They might also observe that each stage consists of one hexagon in addition to six arms of squares.

When the objective is for students to identify a functional relationship—or rule—it is

FIGURE 1

Building an input/output table can limit students' thinking about growing patterns, such as the Expanding Hexagon pattern below.



Stage 1 Stage 2 Stage 3

TABLE 1

Two-column data for the Expanding Hexagon pattern show the independent variable (input) as the *stage number* and the dependent variable (output) as the *total number of pattern blocks*.

Stage number	Total no. of pattern blocks
1	7
2	13
3	19
4	

not uncommon for teachers to extract numerical data from the pattern and use it to build a two-column table. **Table 1** identifies the independent variable (sometimes called *input*) as "stage number" and the dependent variable (sometimes called *output*) as "total number of pattern blocks." The functional relationship that we hope students will identify is the mathematical expression that represents the relationship between the stage number and the total number of pattern blocks.

Unfortunately, once this table is produced, the geometric pattern is frequently ignored. Instead of using what they have noticed about the growing pattern for clues to the functional relationship, students attend to numerical patterns in the two-column table. The most commonly identified numerical pattern is the constant difference—the increase of six pattern blocks as one looks down the right-hand column. Using this constant difference, students add to the previous total to get the number of

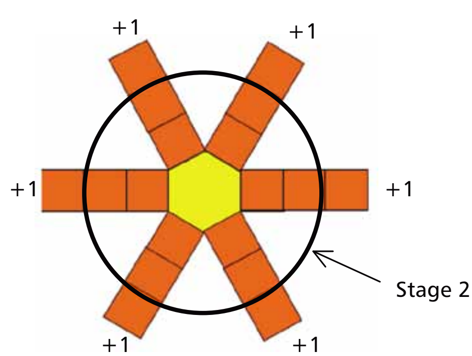
TABLE 2

A series of questions can help students focus on the figures in a pattern sequence (Lee and Freiman 2006). Friel and Markworth (2009, p. 28) modified the questions for this problem-solving process for geometric growing pattern tasks.

Process for geometric growing patterns	
Phase 1: *Reasoning figurally* using the visual characteristics of the geometric pattern task	1. How many different patterns can you see in this drawing? a. How would you draw the next stage? b. How would you draw the 10th stage? c. How would you draw the 58th stage? d. How would you tell someone how to draw any stage at all?
Phase 2: *Developing numerical relationships* to generalize a function	1. How many pattern blocks does it take to make the 10th stage, the 58th stage, or the 100th stage? 2. How many pattern blocks does it take to make the *n*th stage?
Phase 3: *Extending pattern analysis*	1. Which stage has exactly 100 pattern blocks in it? What about 50 square tiles? 2. Can you create a pattern problem for the class?

FIGURE 2

In this diagram of the recursive growth in the Expanding Hexagon pattern, stage 2 is circled within stage 3, and six squares have been identified as the change from stage 2.



pattern blocks needed for future stages.

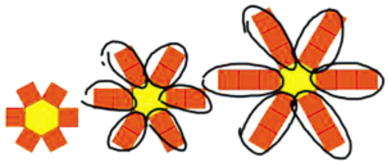
This approach can limit students' thinking. The constant difference represents how the total number of pattern blocks in the pattern changes from one stage to the next, which focuses students' attention on a *recursive* relationship. This is useful for tasks that require *near generalization*; in this case, students would be asked for near stage numbers, such as the number of blocks needed for stage 4, stage 5, or stage 10. For tasks that require *far generalization* (e.g., stage 45 or stage 100), a recursive relationship is inefficient. Although some students are willing to persist in this way (I have had students who have extended a two-column table through stage 100), helping them refocus on a more efficient strategy is important.

The identification of an *explicit* relationship—a rule linking the independent variable to the dependent variable—is more powerful for the purposes of generalization and promotes students' functional thinking. In this case, the objective is to have students generate a relationship between the stage number and the total number of pattern blocks in each stage. If students are directed to the data in a two-column table, they are left to guess the explicit rule or use algorithmic approaches to derive it, neither of which capitalize on the potential of using geometric growing patterns to explore functional relationships.

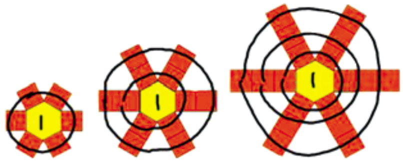
FIGURE 3

A SMART Board™ screen captures various representations of the Expanding Hexagon patterns.

(a) Markworth anticipated students' second way of seeing.



(b) But she did not anticipate a third way:



Promoting functional thinking

Three ways to help students develop functional thinking with geometric growing patterns include (1) promoting various ways of seeing, (2) identifying corresponding ways of counting, and (3) using a three-column table.

1. Ways of seeing

Figural reasoning, or a focus on the visual nature of the pattern itself, leads to generalizations that are more accurate, increased sense making, and better justifications (Lannin, Barker, and Townsend 2006; Rivera and Becker 2005). Lee and Freiman (2006) have identified a series of recommended questions that persuade students to focus on the figures in the pattern sequence. Friel and Markworth (2009) used these questions to elaborate on a three-phase problem-solving process (see **table 2**). The first phase involves questions that promote figural reasoning and typically begin with "How would you draw…?" to focus student thinking on the visual nature of the pattern. The second phase transitions to questions beginning with "How many…?" Students use the figural reasoning developed in the first phase of the process to answer these questions.

In my work with sixth graders, students were to first articulate a *way of seeing* each growing pattern. When questions progressed to phase 2, students were consistently prompted to justify their numerical calculations with their way of seeing the pattern. Surprisingly, three different ways of seeing the Expanding Hexagon pattern emerged during class discussion.

First, students were generally quick to note the recursive relationship in the Expanding Hexagon pattern—that six squares are added to the figure as it increases from stage to stage (see **fig. 2**). Using this way of seeing to identify stage 10 would be difficult; to visualize or calculate stage 10, students would need to know stage 9 first.

I anticipated the second way of seeing, in which the number of squares in each arm of the figures corresponds to the stage number (see **fig. 3a**). Students who identified this way of seeing recognized six groups of square tiles extending from the sides of the central hexagon, rather than a recursive increase in the

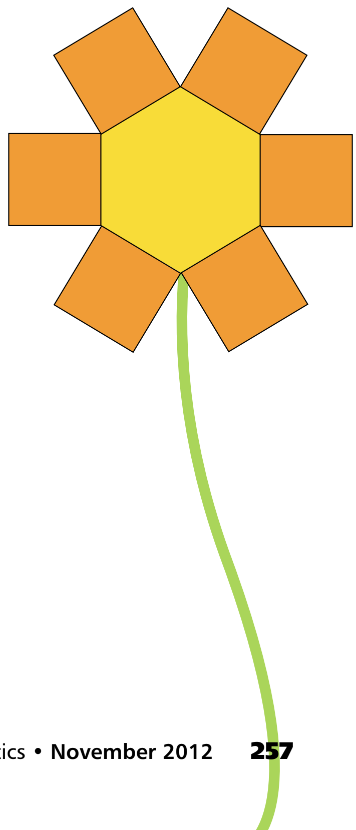


FIGURE 4

Looking at a single stage of a growing pattern is useful for helping students understand that a way of *seeing* is a way of *counting.* See Moss and Beatty (2006) for another version of this task.

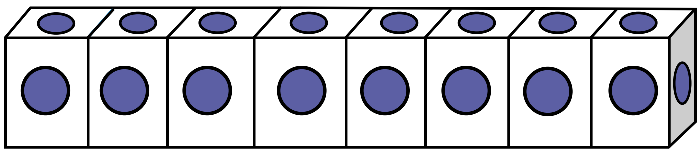


FIGURE 5

In response to being asked to identify the way of seeing that corresponded to the calculations—without counting one by one—students generated multiple ways of counting the stickers on the cubes. This shows eight groups of four stickers with an additional two stickers exposed at the ends: 8 × 4 + 2.

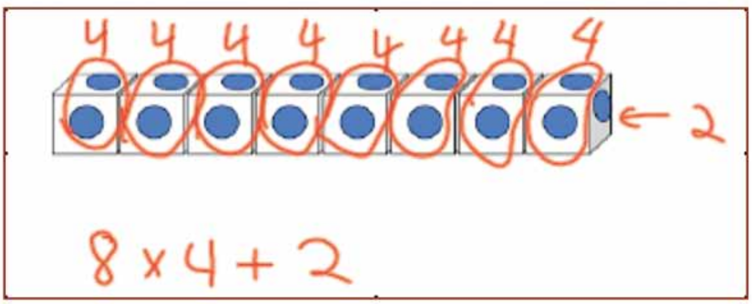
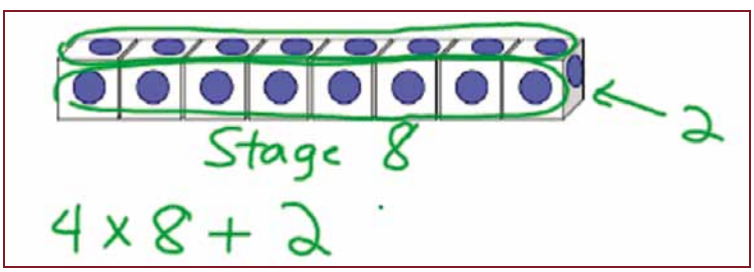


FIGURE 6

Here is a way of seeing four groups of eight stickers with an additional two stickers exposed at the ends: 4 × 8 + 2.



number of tiles from stage to stage. The central hexagon does not change as the pattern grows. (Note: Although it should not be required or suggested at this point in working with a pattern, this way of seeing results in a functional relationship of $T = 6 \times n + 1$, in which T equals the total number of pattern blocks and n equals the stage number.)

I did not anticipate the third way of seeing this geometric pattern (see **fig. 3b**) in which students recognized the single hexagon in the center of each figure. The squares, in contrast, were grouped into concentric circles of six, with the number of circles corresponding to the stage number. Therefore, the stage number indicated how many circles of six squares were arranged around the central hexagon. (Note: This way of seeing results in an equivalent functional relationship of $T = n \times 6 + 1$.)

It is possible that students will identify with the ways of seeing shown in **figure 3** in a more recursive way. In **figure 3a**, a student may describe the pattern as having six arms in which the number of squares increases by one each time. In **figure 3b**, a student may describe the pattern as having concentric circles of six squares in each, such that a circle of squares is added from stage to stage. Representation and discourse can be quite powerful when this occurs. Students' ability to see the correspondence of the stage number to each figure can be influenced by how the teacher records the way of seeing for the class. The representations in **figure 3** enable a more explicit connection between the stage number and the way of seeing. Additionally, emphasis on the groups and the number of groups can support a transition to an explicit relationship: What do you see in the six groups? or How many circles of six squares do you see in each stage?

As students work with various patterns, they should be encouraged to attend carefully and closely to various ways of seeing. Some of their responses will be surprising and unanticipated. Valuing multiple ways of seeing is important because students need to evaluate them as they work toward numerical and functional relationships.

2. Ways of counting

Despite support and encouragement for multiple ways of seeing, students may still wonder,

"What's the point?" They may not realize that a way of seeing how a growing pattern is structured provides an efficient *way of counting* for finding the dependent variable in a functional relationship. In the Expanding Hexagon pattern, one could simply count the number of pattern blocks that are found in stages 1, 2, and 3. However, a way of seeing provides a meaningful way of counting that helps in making near and far generalizations (Mason 1996).

For this reason, looking at a single stage of a growing pattern is useful for helping students see that a *way of seeing* is a *way of counting*. One lesson supplied an aha moment for several of the students I worked with. In this lesson, I showed a picture (see **fig. 4**), gave students manipulatives, and asked how many stickers would be exposed if eight cubes, with a sticker on each face, were connected end to end. Further, I asked if they could determine the answer without counting the stickers one by one.

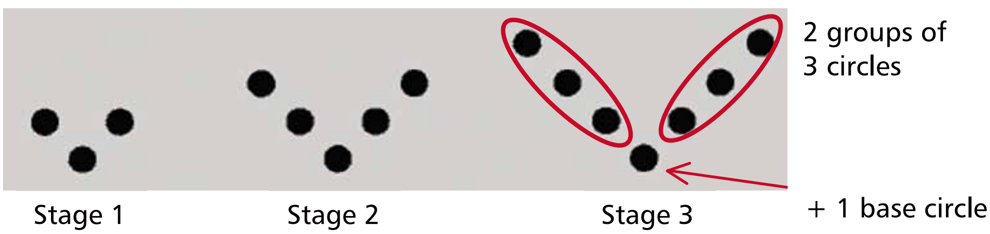
The youngsters generated multiple ways of counting the stickers. For each approach, I asked the student to identify the way of seeing that corresponded to the calculations. **Figure 5** shows eight groups of four stickers with an additional two stickers exposed at the ends. (This representation shows only two stickers on each cube. In accounting for the unseen stickers, students seemed comfortable with labeling these as groups of four.) The numerical calculation for this way of seeing is $8 \times 4 + 2$. Alternatively, **figure 6** shows four groups of eight stickers with an additional two stickers exposed at the ends. (Again, students seemed to recognize that the groups shown on the screen were duplicated on the actual figure.) The numerical calculation for this way of seeing is $4 \times 8 + 2$.

After ways of seeing and counting have been shared, **figures 5** and **6** can be identified as stage 8 of a growing pattern, because eight cubes are connected. Students can be asked to use one of the ways of counting to calculate the number of exposed stickers if seven cubes (or any other stage number) were connected end to end. These questions help students consider which aspects of the calculation stay the same (the multiplier of 4; the constant of + 2) and what changes (the number of cubes, or the stage number). Connecting this to the way of seeing provides support for students' functional thinking as they make an explicit connection between the stage number, the way of seeing, and the way of counting. In later work with algebra, these connections will be generalized to $T = n \times 4 + 2$ or $T = 4 \times n + 2$ in which T equals the total number of stickers and n equals the stage number.

Because students may have difficulty identifying a way of seeing and counting the growing pattern within the first two stages, analysis of a more advanced stage can be useful in any geometric pattern. For example, a structure is not clear in the first stage of the growing pattern in **figure 7**. When considering stage 3, at least one way of seeing becomes evident: identifying two groups of three circles extending from a base circle, or $2 \times 3 + 1$. The stage number corresponds to the number of circles in each group; the functional relationship would be $2 \times n + 1$. Looking back at the previous two stages confirms that this way of seeing works for all three stages of the pattern.

FIGURE 7

Students may have difficulty identifying a way of seeing and counting the growing pattern within the first two stages, but considering stage 3 may give them at least one way of seeing (V pattern, NCRMSE and Freudenthal Institute 1998).



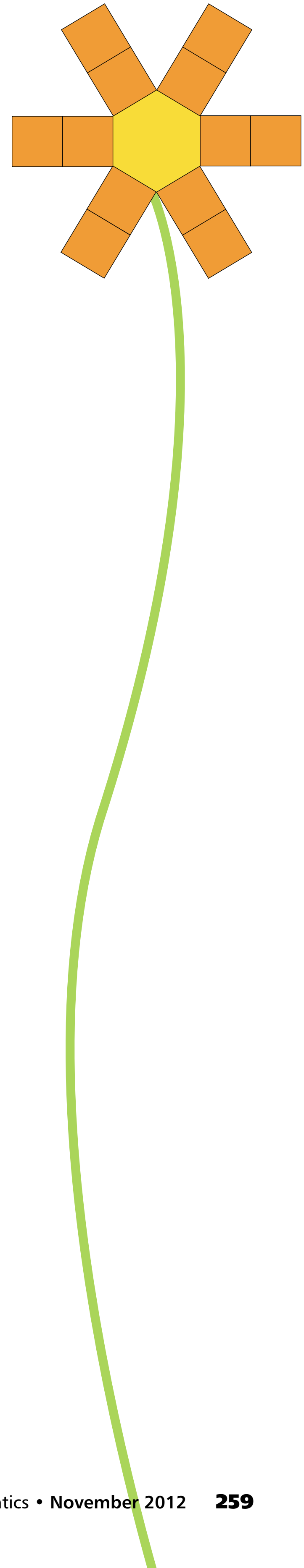


TABLE 3

A three-column table represents the way of seeing shown in **figure 3**.

Representation of the Expanding Hexagon

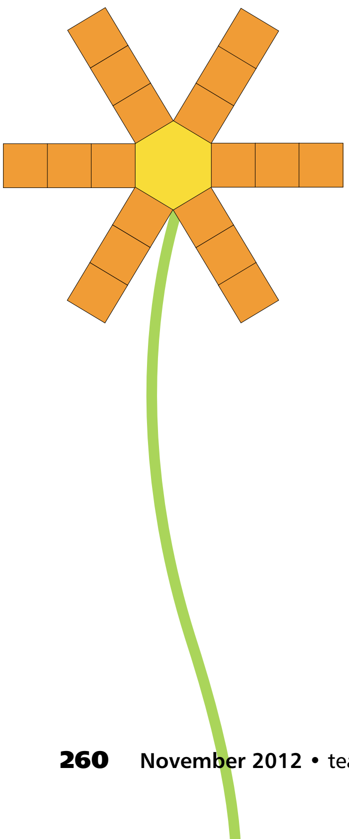
Stage number	My thinking	Total no. of pattern blocks
1	$6 \times 1 + 1$	7
2	$6 \times 2 + 1$	13
3	$6 \times 3 + 1$	19
4	$6 \times 4 + 1$	25

TABLE 4

When students are ready to proceed to variables, have them use a three-column table to generate a symbolic rule.

Generating a Symbolic Rule

Stage number	My thinking	Total no. of pattern blocks
1	$6 \times 1 + 1$	7
2	$6 \times 2 + 1$	13
3	$6 \times 3 + 1$	19
4	$6 \times 4 + 1$	25
n	$6 \times n + 1$	



If students struggle to identify a way of seeing that connects to an efficient way of counting, encourage them to look at the third stage of the pattern: What do you notice about stage 3? Can you find a way of counting the number of circles in stage 3 that is related to the stage number? Such questions encourage students to examine the structure of the geometric growing pattern to identify a way of seeing that is related to the stage number.

3. A three-column table

As previously noted, when students use a two-column table to identify patterns, they are drawn to the vertical relationship in the right-hand column. This promotes recursive thinking, because it highlights the progression of the dependent variable from one stage to the next. Advancing functional thinking requires attention to the horizontal relationship between the independent and dependent variables. Using a three-column table, with the calculation of the dependent variable in the middle column, better supports functional thinking (Friel and Markworth 2009). The three-column table is used frequently in algebra instructional materials by Math Solutions for grades 3–5 and 6–8 (Lawrence and Hennessy 2002; Wickett, Kharas, and Burns 2002) and appears in various forms in other research (Driscoll 1999; Carraher, Martinez, and Schliemann 2008).

The three-column table offers students a method of translating a geometric way of seeing into a numerical way of counting, which can be generalized into a functional relationship using words, symbols, and eventually variables. Let's return to the Expanding Hexagon pattern as an introduction to the three-column table. Recall that one way of seeing this pattern is as six legs of squares, each comprising the stage number of squares, around an interior hexagon (see **fig. 3**). This way of counting the total number of pattern blocks is shown in the center column of **table 3**. At each stage, six is multiplied by the stage number to represent six legs of squares. One is added to this product to represent the center hexagon at each stage.

Connecting the numerical calculations in the center column to the way of seeing that has been identified is crucial. For example, in this case, ask students what the 6 represents in the center column and why it is multiplied by the 1, 2, and 3, respectively. Also ask students what the + 1 represents in the pattern. **Table 3** can be used for near generalization as students calculate the fourth, fifth, and tenth stages. However, these calculations should be justified geometrically with the way of seeing the pattern.

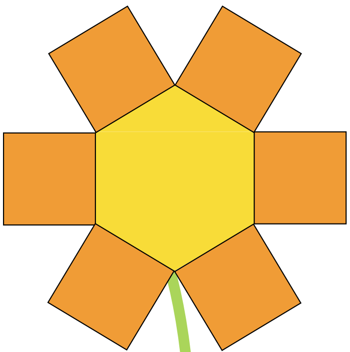
The three-column table is an effective tool for helping students connect the independent variable to the dependent variable in a functional relationship. The center column provides a location for them to translate their way of seeing

to a way of counting. Both the recording of this and the observation of patterns within the table furnish meaningful access for the identification of an explicit functional relationship.

These observations can be effectively transformed into generalized words: six times the stage number plus one. Students will likely be comfortable and relieved to write this in a semisymbolic form: 6 × stage number + 1. When they seem ready to proceed to variables, the three-column table is an effective way to introduce them, as well. Instead of placing a numerical value in the stage number column, ask what might be done if we do not know the stage number but call it n for any stage number. Students will be able to substitute *stage number* with a variable and keep other aspects of the calculation the same (see **table 4**). Multiple three-column tables for a geometric growing pattern, each derived from a different way of seeing, can elicit valuable discussions regarding the equivalency of the numerical expressions (e.g., $6 \times 3 + 1$ and $3 \times 6 + 1$) and the generalized algebraic expressions (e.g., $6 \times n + 1$ and $n \times 6 + 1$). Such discussions can make important connections to prior knowledge, such as the application of the commutative property of multiplication. They can also provide a foundation for future work in algebra, such as simplifying algebraic expressions.

Connecting geometric growing patterns to functional relationships

Children are introduced to functional relationships in elementary and middle school through function machines and games like Guess My Rule. Although these are valuable learning experiences, used alone, they may encourage the notion that functional relationships must be guessed or that they have no meaning. Working with functional relationships through geometric growing patterns has the potential to reveal meaning for the numerical calculations.



Students can connect geometric growing patterns to functional relationships, thereby making sense of the operations that transform the independent variable to the dependent variable.

These three teaching suggestions promote meaningful, structured learning experiences for students working with geometric growing patterns. Students have opportunities to transition from a concrete way of seeing the growing pattern to a numerical representation. Connecting these with a three-column table enables students to make sense of the explicit functional relationship. Exploring geometric growing patterns is a powerful approach for supporting students' development of functional thinking. If we extend ways of seeing to ways of counting and make meaningful connections between the two, our students will develop understanding that better prepares them for more advanced and abstract work with functions.

REFERENCES

Carraher, David W., Mara V. Martinez, and Analucia D. Schliemann. 2008. "Early Algebra and Mathematical Generalization." *ZDM Mathematics Education* 40:3–22.

Driscoll, Mark. 1999. *Fostering Algebraic Thinking: A Guide for Teachers 6–10*. Portsmouth, NH: Heinemann.

Friel, Susan N., and Kimberly A. Markworth. 2009. "A Framework for Analyzing Geometric Pattern Tasks." *Mathematics Teaching in the Middle School* 15 (August): 24–33.

Kieran, Carolyn. 2007. "Learning and Teaching Algebra at the Middle School through College Levels: Building Meaning for Symbols and Their Manipulation." In *Second Handbook of Research on Mathematics Teaching and Learning*, edited by Frank K. Lester Jr., pp. 707–62. Charlotte, NC: Information Age Publishing.

Lannin, John K., David D. Barker, and Brian E. Townsend. 2006. "Recursive and Explicit Rules: How Can We Build Student Algebraic Understanding?" *Journal of Mathematical Behavior* 25 (4): 299–317.

Lawrence, Ann, and Charlie Hennessy. 2002. *Lessons for Algebraic Thinking: Grades 6–8*. Sausalito, CA: Math Solutions Publications.

Lee, Lesley, and Viktor Freiman. 2006. "Developing Algebraic Thinking through Pattern Exploration." *Mathematics Teaching in the Middle School* 11 (May): 428–33.

Mason, John. 1996. "Expressing Generality and Roots of Algebra." In *Approaches to Algebra: Perspectives for Research and Teaching*, edited by Nadine Bednarz, Carolyn Kieran, and Lesley Lee, pp. 65–86. Dordrecht, Netherlands: Kluwer Academic Publishers.

Moss, Joan, and Ruth Beatty. 2006. "Knowledge Building in Mathematics: Supporting Collaborative Learning in Pattern Problems." *International Journal of Computer-Supported Collaborative Learning* 1 (4): 441–65.

Moss, Joan, Ruth Beatty, Susan London McNab, and Janet Eisenband. 2006. "The Potential of Geometric Sequences to Foster Young Students' Ability to Generalize in Mathematics." http://www.brookings.edu/brown/~/media/Files/Centers/bcep/AlgebraicReasoningConferenceMoss.pdf.

National Center for Research in Mathematical Sciences Education (NCRMSE) and Freudenthal Institute. 1998. *Mathematics in Context: Patterns and Figures*. Chicago: Encyclopedia Britannica Educational Corporation.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Rivera, Ferdinand D., and Joanne Rossi Becker. 2005. "Figural and Numerical Modes of Generalizing in Algebra." *Mathematics Teaching in the Middle School* 11 (November): 198–203.

———. 2009. "Algebraic Reasoning through Patterns." *Mathematics Teaching in the Middle School* 15 (November): 213–21.

Smith, Erick. 2008. "Representational Thinking as a Framework for Introducing Functions in the Elementary Curriculum." In *Algebra in the Early Grades*, edited by James J. Kaput, David W. Carraher, and Maria L. Blanton, pp. 133–60. New York: Lawrence Erlbaum Associates.

Wickett, Maryann, Katharine Kharas, and Marilyn Burns. 2009. *Lessons for Algebraic Thinking: Grades 3–5*. Sausalito, CA: Math Solutions Publications.

LAUREN McCLANAHAN

Kimberly A. Markworth, **kimberly.markworth@wwu.edu**, is an assistant professor of mathematics education at Western Washington University. She is interested in developing tasks that promote children's algebraic reasoning at elementary and middle school levels.

NCTM 2012

Regional Conferences & Expositions

DALLAS, TX | OCTOBER 10–12
HARTFORD, CT | OCTOBER 24–26
CHICAGO, IL | NOVEMBER 28–30

Join the Best & Brightest Math Educators

NCTM's Regional Conferences have the professional development opportunities you need to help your students succeed. Join the best and the brightest and immerse yourself in the latest topics in math education. By attending you and your colleagues will:

- Learn more about and test the latest **technology**
- Discover new and effective **intervention** methods
- Learn practices central to teaching the **Common Core State Standards**
- Refine your **assessment** techniques
- And more!

Whether you're a classroom teacher, coach, administrator, preservice teacher, or math specialist, this conference has something for you.

NCTM®
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
(800) 235-7566 | WWW.NCTM.ORG

Visit **www.nctm.org/regionals** to learn more and register.